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An Explicit Iteration for Asymptotic Centers of Orbits of Nonexpansive Mappings in Banach Spaces

NORIMICHI HIRANO

Department of Mathematics, Faculty of Engineering, Yokohama National University, Yokohama 156, Japan

AND

KAZUO KIDO AND WATARU TAKAHASHI

Department of Information Science, Tokyo Institute of Technology, Tokyo 152, Japan

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INTRODUCTION

Let E be a real Banach space, and T a nonexpansive mapping of E into itself. For $x \in E$, an asymptotic center of the orbit $\{T^i x\}_{i=0}^{\infty}$ (with respect to E) is an element z of E such that $\overline{\lim}_i \|T^i x - z\| \leqslant \overline{\lim}_i \|T^i x - y\|$ for all $y \in E$.

Brézis and Browder [4] studied an iteration to obtain the asmptotic center of an orbit of T as a nonlinear ergodic theorem. That is, the asymptotic center is approximated in the weak topology of E by the unique minimum points of appropriate convex functions f_n, $n = 1, 2, \ldots$. However, it is an implicit iteration. On the other hand, many authors studied explicit iterations to obtain fixed points of nonexpansive mappings; for example, double iterations, mean ergodic theorems, mean value iterations, and so on; see [1, 5, 6, 11, 13, 14] and the references mentioned there. However, it seems that explicit iterations to obtain asymptotic centers have never been carried out in Banach spaces (cf. [12]).

In this paper we study an explicit iteration scheme converging weakly to the asymptotic center of an orbit of a nonexpansive mapping T (with fixed points) in a uniformly convex and uniformly smooth Banach space with a weakly continuous duality mapping. In particular, we obtain an iteration scheme which is calculated from finite number of iterates $\{x, Tx, \ldots, T^n x\}$ at each time.

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Our main result is Theorem 2. That is, we find a sequence $\{y_n\}$ converging to the asymptotic center of a given orbit $\{T^i x\}$ of T, where y_n approximates the minimum point of f_n. This result extends the main result of [4] partially. Such a y_n is constructed explicitly from $\{x, Tx, ..., T^n x\}$ by using Theorem 1 and Remark 1 in this paper. This iteration is a kind of gradient method, which is adequate for the condition of Theorem 2; confer [7] for the gradient method for C^1 functionals. Our proofs are simple on account of using Banach limits.

1. Preliminaries

Let $B(N)$ be the Banach space of all bounded real valued functions defined on $N = \{0, 1, 2, ...\}$ with the supremum norm. An element μ of the dual space $B(N)^*$ of $B(N)$ is said to be a *mean* on N if $\mu(1) = 1 = \|\mu\|$. Furthermore a mean μ is said to be an *invariant mean* (or *Banach limit*) if μ satisfies that $\mu(r_n f) = \mu(f)$ for all $n \in N$ and $f \in B(N)$, where $r_n f(m) = f(m+n)$ for $m, n \in N$.

Next, let A be a (multivalued) mapping from a real Banach space E into E^*. A is said to be *monotone* if A satisfies that $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$ for all $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, where $D(A)$ denotes the domain of A. A monotone mapping A is said to be *maximal* if A has no monotone extension. Let f be a proper convex lower semi-continuous function of E into $(-\infty, \infty]$. Then we define the *subdifferential* ∂f of f by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geqslant \langle x^*, y - x \rangle \text{ for all } y \in E\}$$

for all $x \in E$. It is well known that ∂f is a maximal monotone operator from E into E^*. Let h, called a *gauge function*, be a strictly increasing and continuous function from $[0, \infty)$ to $[0, \infty)$ such that $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$. Then we denote by J_h the *duality mapping* with gauge function h defined by

$$J_h(x) = \{x^* \in E^* : \langle x^*, x \rangle = \|x\| \cdot \|x^*\|, \|x^*\| = h(\|x\|)\} \qquad \text{for} \quad x \in E.$$

J_h is known to be the subdifferential of the mapping $\Phi(\|\cdot\|)$, where $\Phi(t) = \int_0^t h(s)\, ds$, $t \geqslant 0$. Furthermore suppose that E^* is uniformly convex. Then J_h is single valued and uniformly continuous on bounded subsets of E. Therefore for a bounded subset D of E and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|J_h u - J_h v\| \leqslant \varepsilon \qquad \text{for all } u, v \in D: \|u - v\| \leqslant \delta. \tag{1.1}$$

Then we define a positive number $\delta_D(\varepsilon)=\sup\{\delta>0: \delta \text{ satisfies } (1.1)\}$. Since $\delta_D(\varepsilon)\to 0$ as $\varepsilon\to 0$, we also define $\delta_D(0)=0$.

Finally, let $\{y_i\}$ be a bounded sequence in E. For a mean μ on N we define $f_\mu(x)=\int \Phi(\|y_i-x\|)\,d\mu(i)$ for all $x\in E$, where the right term denotes the value of μ at the function $\psi(i)=\Phi(\|y_i-x\|)$, $i\in N$. Then, as in [4], (1) f_μ is a lower semi-continuous convex function of E into $[0,\infty)$; (2) f_μ has a unique minimum point $y_\mu\in E$ (i.e., $f_\mu(y_\mu)=\min\{f_\mu(x): x\in E\}$); (3) $\langle \partial f_\mu(x), z\rangle=\int \langle J_h(y_i-x), z\rangle\,d\mu(i)$ for all $x, z\in E$; (4) $f_\mu(z)\to\infty$ as $\|z\|\to\infty$. Similarly we define $g(x)=\limsup_i \|y_i-x\|$ for all $x\in E$. Then we say $\bar{y}\in E$ is an *asymptotic center* of $\{y_i\}$ (with respect to E) when $\bar{y}$ is a minimum point of g in E (ie., $g(\bar{y})=\min\{g(x): x\in E\}$). If E is uniformly convex, then there exists a unique asymptotic center for each bounded sequence in E (cf. [9]).

2. Main Results

Throughout this section we assume that E is a real uniformly convex and uniformly smooth Banach space. Let $\{y_i\}$ be a bounded sequence in E, μ a mean on N, and h a gauge function.

First we consider the following explicit iteration scheme (confer, for example, [7]):

$$\begin{aligned} &x_0\in E,\\ &x_{n+1}=x_n-\lambda_n J_k^* \partial f_\mu(x_n), \qquad n=0,1,2,..., \end{aligned} \tag{2.1}$$

where $\lambda_n>0$ and J_k^* is the single valued duality mapping of E^* into E with gauge function $k=h^{-1}$.

THEOREM 1. *Let D be a bounded subset of E. Assume that $\{x_n\}$ satisfies (2.1), $y_i-x_n\in D$ for $i, n=0,1,2,...,$ and*

$$\alpha\cdot\delta_D(\|\partial f_\mu(x_n)\|/2)\leqslant \|x_{n+1}-x_n\|\leqslant \delta_D(\|\partial f_\mu(x_n)\|/2), \qquad n=0,1,... \tag{2.2}$$

for some $\alpha>0$. Then x_n converges weakly to y_μ as $n\to\infty$, where y_μ is a unique minimum point of f_μ defined under preliminaries.

Proof. Since f_μ is lower semi-continuous and convex,

$$\begin{aligned} f_\mu(x_{n+1})-f_\mu(x_n)&\leqslant \langle \partial f_\mu(x_{n+1}), x_{n+1}-x_n\rangle\\ &=-\lambda_n\langle \partial f_\mu(x_{n+1}), J_k^*\partial f_\mu(x_n)\rangle. \end{aligned}$$

On the other hand, using (2.2),

$$\begin{aligned}
&|\|\partial f_\mu(x_n)\| \cdot k(\|\partial f_\mu(x_n)\|) - \langle \partial f_\mu(x_{n+1}), J_k^* \partial f_\mu(x_n)\rangle| \\
&\quad = |\langle \partial f_\mu(x_n) - \partial f_\mu(x_{n+1}), J_k^* \partial f_\mu(x_n)\rangle| \\
&\quad \leqslant k(\|\partial f_\mu(x_n)\|) \cdot \|\partial f_\mu(x_n) - \partial f_\mu(x_{n+1})\| \\
&\quad \leqslant k(\|\partial f_\mu(x_n)\|) \cdot \sup_i \|J_h(y_i - x_n) - J_h(y_i - x_{n+1})\| \\
&\quad \leqslant k(\|\partial f_\mu(x_n)\|) \cdot \|\partial f_\mu(x_n)\|/2.
\end{aligned}$$

Therefore we obtain that

$$f_\mu(x_{n+1}) - f_\mu(x_n) \leqslant -\lambda_n \cdot k(\|\partial f_\mu(x_n)\|) \cdot \|\partial f_\mu(x_n)\|/2 \tag{2.3}$$

for every $n = 0, 1, 2, \ldots$. Thus $\{f_\mu(x_n)\}$ is a non-increasing sequence of non-negative numbers and converges to some $c \geqslant 0$. Then, since $\lambda_n \cdot k(\|\partial f_\mu(x_n)\|) \cdot \|\partial f_\mu(x_n)\| \leqslant 2(f_\mu(x_n) - f_\mu(x_{n+1}))$, $\lambda_n \cdot k(\|\partial f_\mu(x_n)\|) \cdot \|\partial f_\mu(x_n)\|$ converges to 0 as $n \to \infty$.

Now we prove $\partial f_\mu(x_n) \to 0$ as $n \to \infty$ by contradiction. Assume that there exist $M > 0$ and a subsequence $\{x_{n(m)}\}$ of $\{x_n\}$ such that $\|\partial f_\mu(x_{n(m)})\| \geqslant M$ for all $m \geqslant 0$. Then we have

$$\begin{aligned}
&\lambda_{n(m)} \cdot k(\|\partial f_\mu(x_{n(m)})\|) \cdot \|\partial f_\mu(x_{n(m)})\| \geqslant M \cdot \|x_{n(m)+1} - x_{n(m)}\| \\
&\quad \geqslant M \cdot \alpha \cdot \delta_D(\|\partial f_\mu(x_{n(m)})\|/2) \\
&\quad \geqslant M \cdot \alpha \cdot \delta_D(M/2) > 0.
\end{aligned}$$

Since $\lambda_{n(m)} \cdot k(\|\partial f_\mu(x_{n(m)})\|) \cdot \|\partial f_\mu(x_{n(m)})\| \to 0$ as $m \to \infty$, this is a contradiction.

Lastly we show that x_n converges weakly to y_μ as $n \to \infty$. Since $\{x_n\}$ is bounded, taking the subsequence if necessary, we may assume that x_n converges weakly to ω and $\partial f_\mu(x_n) \to 0$ as $n \to \infty$. Then, since ∂f_μ is a maximal monotone operator from E into E^*, we obtain $0 \in \partial f_\mu(\omega)$. The uniqueness of the minimum point of f_μ yields $\omega = y_\mu$. Therefore the conclusion follows.

Let T be a nonexpansive mapping of E into itself with fixed points. For the remainder of this paper, we consider $\{y_i\} = \{T^i x\}$ for a fixed $x \in E$.

Remark 1. For any $x_0 \in E$, there does exist a bounded subset D of E and a sequence $\{x_n\}$ satisfying the whole condition of Theorem 1 for $\{y_i\} = \{T^i x\}$. Indeed, we have $f_\mu(x_0) \leqslant \sup_i \Phi(\|T^i x - x_0\|) \leqslant \sup_i \Phi(\|T^i x - \bar{x}\| + \|\bar{x} - x_0\|) \leqslant \Phi(\|x - \bar{x}\| + \|\bar{x} - x_0\|) \equiv r < \infty$, where $\bar{x}$ is a fixed point of T. Since $f_\mu(z) \to \infty$ as $\|z\| \to \infty$, $D_1 \equiv f_\mu^{-1}(-\infty, r]$ is bounded. Therefore there exists $R > 0$ such that $D_1 \cup \{T^i x\}_{i=0}^{\infty} \subset B_R[0]$, where $B_R[0]$ denotes the closed ball with center 0 and radius R. Here let $R' = \max\{R, k(h(2R)/2)\}$, $D \equiv B_{3R'}[0]$, and suppose that $x_n \in D_1$. Select $\lambda_n > 0$ so that $\|x_{n+1} - x_n\| = \min\{R', \delta_D(\|\partial f_\mu(x_n)\|/2)\}$. Then $x_{n+1} \in B_{2R'}[0]$, $T^i x - x_{n+1} \in D$ for $i = 0, 1, 2, \ldots$, and (2.3) holds. Therefore, we have $f_\mu(x_{n+1}) \leqslant f_\mu(x_n) \leqslant r$ and $x_{n+1} \in D_1$. Since $x_0 \in D_1$, we obtain a sequence $\{x_n\}_{n=0}^{\infty}$ in D inductively. Finally we show the existence of $\alpha > 0$ which satisfies (2.2). For every $n = 0, 1, 2, \ldots$, since $0 \in D$, $\delta_D(\|\partial f_\mu(x_n)\|/2) \leqslant k(\|\partial f_\mu(x_n)\|/2) \leqslant k(\sup_i h(\|T^i x - x_n\|)/2) \leqslant k(h(2R)/2) \leqslant R'$. Therefore, (2.2) holds for $\alpha = 1$.

Let $\bar{y}$ be the asymptotic center of $\{T^i x : i = 0, 1, 2, \ldots\}$. Then $\bar{y}$ is a fixed point of T. Furthermore $\bar{y} = y_\mu$ for every Banach limit μ. Indeed, since y_μ and $\bar{y}$ are fixed points of T, we have $\limsup_i \Phi(\|T^i x - y_\mu\|) = \int \Phi(\|T^i x - y_\mu\|)\, d\mu(i) \leqslant \int \Phi(\|T^i x - \bar{y}\|)\, d\mu(i) = \limsup_i \Phi(\|T^i x - \bar{y}\|)$. Then, by the strict increasingness of Φ and the uniqueness of the asmptotic center, we obtain $y_\mu \equiv \bar{y}$. Using this fact, we consider iteration schemes for finding the asymptoic center of $\{T^i x\}$. Let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_i, \qquad n = 1, 2, \ldots, \tag{2.4}$$

be *Cesàro means* on N, where δ_i is a mean on N defined by $\delta_i(f) = f(i)$ for all $f \in B(N)$. Then $f_{\mu_n}(z) = (1/n) \sum_{i=0}^{n-1} \Phi(\|T^i x - z\|)$ and $\partial f_{\mu_n}(z) = (1/n) \sum_{i=0}^{n-1} J_h(T^i x - z)$ for all $z \in E$. Fix $n \in N$ and let $z_n \in E$. Define $z_{n+1} \in E$ as follows. From Theorem 1 and Remark 1, there exists a sequence $\{x_{n,m}\}_{m=0}^{\infty}$ which satisfies

$$x_{n,0} = z_n, \tag{2.5}$$

$$x_{n,m+1} = x_{n,m} - \lambda_m J_k^* \partial f_{\mu_{n+1}}(x_{n,m}), \qquad m = 0, 1, 2, \ldots, \tag{2.6}$$

and

$$\|\partial f_{\mu_{n+1}}(x_{n,m})\| \to 0 \qquad \text{as} \quad m \to \infty.$$

In particular, there exists $M_n > 0$ such that

$$\|\partial f_{\mu_{n+1}}(x_{n,m})\| \leqslant 1/(n+1) \qquad \text{for all } m \geqslant M_n. \tag{2.7}$$

Then let

$$z_{n+1} = x_{n,m} \qquad \text{for some } m \geqslant M_n. \tag{2.8}$$

This procedure yields a sequence $\{z_n\}$ inductively from a given initial element $z_0 \in E$. We remark here that z_n, $n = 1, 2, \ldots$, is obtained explicitly from finite numbers of $T^i x$ by (2.6). On the convergence of such z_n, we obtain the following theorem.

THEOREM 2. *Let μ_n, $n=1, 2, \ldots$, be Cesàro means. Let $\{z_n\}$ be a sequence in E such that*

$$\|\partial f_{\mu_n}(z_n)\| \to 0 \qquad as \quad n \to \infty. \tag{2.9}$$

If the duality mapping J_h is weakly continuous, then z_n converges weakly to the asymptotic center $\bar{y}$ of $\{T^i x\}$ as $n \to \infty$.

Proof. First we show that $\{z_n\}$ is bounded. If $\{z_n\}$ is not bounded, then there exists a subsequence $\{z_{n(m)}\}$ of $\{z_n\}$ such that $\|z_{n(m)}\| \geqslant m$. Since $F(T) \neq \phi$, $\{T^i x\}$ is bounded, say $\|T^i x\| \leqslant M$ for $i=0, 1, 2, \ldots$. Now let $\varepsilon > 0$ arbitrarily. Then, by the uniform continuity of J_h on bounded subsets of E, we have

$$\|J_h(-z_{n(m)}/\|z_{n(m)}\|) - J_h((T^i x - z_{n(m)})/\|z_{n(m)}\|)\| < \varepsilon$$

for $i=0, 1, 2, \ldots$, when m is sufficiently large. Since $J_h(rx) = (h(\|rx\|)/h(\|x\|))J_h(x)$ for every $r > 0$ and $0 \neq x \in E$, the above inequality implies

$$\begin{aligned}
&\|\partial f_{\mu_{n(m)}}(z_{n(m)})\| \cdot \|z_{n(m)}\| \\
&\quad \geqslant \langle \partial f_{\mu_{n(m)}}(z_{n(m)}), -z_{n(m)} \rangle = \int \langle J_h(T^i x - z_{n(m)}), -z_{n(m)} \rangle \, d\mu_{n(m)}(i) \\
&\quad \geqslant \int \|z_{n(m)}\| \cdot (h(1) - \varepsilon) \cdot h(\|T^i x - z_{n(m)}\|)/h(\|T^i x - z_{n(m)}\|/\|z_{n(m)}\|) \, d\mu_{n(m)}(i).
\end{aligned}$$

Therefore we obtain

$$\|\partial f_{\mu_{n(m)}}(z_{n(m)})\| \geqslant \frac{h(1) - \varepsilon}{h(1 + \varepsilon)} h(\|z_{n(m)}\| - M)$$

for sufficiently large m. Since $\|z_{n(m)}\| \geqslant m$, this contradicts (2.9).

So taking the subnet if necessary, we may assume that z_n converges weakly to some $\omega \in E$. Then we must only prove that $\omega = \bar{y}$. Again taking the subnet in $B(N)^*$, we may assume that $\mu_{n(\alpha)}$ converges to a Banach limit μ on N in the weak* topology of $B(N)^*$. Then clearly we have that $z_{n(\alpha)}$ converges weakly to ω and that $\|\partial f_{\mu_{n(\alpha)}}(z_{n(\alpha)})\|$ converges to 0. Fix $0 \neq y \in E$ and define $g_\alpha^y(i) = \langle J_h(T^i x - z_{n(\alpha)}), y \rangle$ and $g^y(i) = \langle J_h(T^i x - \omega), y \rangle$ for $i=0, 1, 2, \ldots$. Then, since J_h is uniformly weakly continuous on bounded subsets of E, $g_\alpha^y(i)$ converges to $g^y(i)$ uniformly in i as α takes infinity.

Therefore, taking α sufficiently large for any fixed $\varepsilon > 0$, we may assume that

$$|\mu(g^y) - \mu_{n(\alpha)}(g^y)| < \varepsilon,$$
$$\sup_i |g_\alpha^y(i) - g^y(i)| < \varepsilon,$$

and

$$\|\partial f_{\mu_{n(\alpha)}}(z_{n(\alpha)})\| < \varepsilon / \|y\|.$$

Then we have

$$\begin{aligned}
|\mu(g^y)| &\leqslant |\mu(g^y) - \mu_{n(\alpha)}(g^y)| + |\mu_{n(\alpha)}(g^y) - \mu_{n(\alpha)}(g_\alpha^y)| + |\mu_{n(\alpha)}(g_\alpha^y)| \\
&\leqslant \varepsilon + \|\mu_{n(\alpha)}\| \cdot \sup_i |g^y(i) - g_\alpha^y(i)| \\
&\quad + \left| \int \langle J_h(T^i x - z_{n(\alpha)}), y \rangle \, d\mu_{n(\alpha)}(i) \right| \\
&\leqslant 2\varepsilon + |\langle \partial f_{\mu_{n(\alpha)}}(z_{n(\alpha)}), y \rangle| \\
&\leqslant 3\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain $\mu(g^y) = 0$ for each $0 \neq y \in E$. That is, $\langle \partial f_\mu(\omega), y \rangle = 0$ for each $y \in E$. Therefore $\omega \in \partial f_\mu^{-1}(0)$. Since μ is a Banach limit, $\partial f_\mu^{-1}(0) = \{\bar{y}\}$, and the proof is completed.

Remark 2. The above theorem is proved for general sequences, that satisfy (2.9), independent of special constructions of them.

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